

PRODUCTIVELY LINDELÖF SPACES OF COUNTABLE TIGHTNESS

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ABSTRACT. Michael asked whether every productively Lindelöf space is powerfully Lindelöf. Building on work of Alster and De la Vega, assuming the Continuum Hypothesis, we show that every productively Lindelöf space of countable tightness is powerfully Lindelöf. This strengthens a result of Tall and Tsaban. The same methods also yield new proofs of results of Arkhangel'skii and Buzyakova. Furthermore, assuming the Continuum Hypothesis, we show that a productively Lindelöf space X is powerfully Lindelöf if every open cover of X^ω admits a point-continuum refinement consisting of basic open sets. This strengthens a result of Burton and Tall. Finally, we show that separation axioms are not relevant to Michael's question: if there exists a counterexample (possibly not even \mathfrak{T}_0), then there exists a regular (actually, zero-dimensional) counterexample.

The research in this article is ultimately motivated by the following well-known question, which is credited to Michael by Alster (see [1]). Recall that a space X is *productively Lindelöf* if $X \times Y$ is Lindelöf for every Lindelöf space Y , and it is *powerfully Lindelöf* if X^ω is Lindelöf. For all other notation and terminology, see Section 1.

Question 1 (Michael). Does productively Lindelöf imply powerfully Lindelöf?

Notice that if X is productively Lindelöf then X^n is Lindelöf for every $n \in \omega$. While, assuming CH, there exists a non-powerfully Lindelöf space X such that X^n is Lindelöf for every $n \in \omega$ (see [7, Example 1.2]), Question 1 remains open under any set-theoretic assumption. The following seems to be the most substantial result on the subject (see [1, Theorem 2]).

Theorem 1 (Alster). *Assume CH. If X is a productively Lindelöf space and $w(X) \leq \mathfrak{c}$ then X is powerfully Lindelöf.*

The technique of elementary submodels has been successfully employed by several authors to establish further consequences of the above theorem. For example, Burton and Tall obtained Theorem 5 below, while Tall and Tsaban obtained the following result (see [9, Theorem 1.4]).

Theorem 2 (Tall, Tsaban). *Assume CH. If X is a productively Lindelöf sequential space then X is powerfully Lindelöf.*

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Continuing in this tradition, we will show that Theorem 2 can be improved by weakening “sequential” to “of countable tightness” (see Theorem 7). Furthermore, we will show that separation axioms are irrelevant to Question 1, Theorem 1 and Theorem 5 (see Corollary 14, Theorem 13 and Corollary 17 respectively). Finally, we will obtain a strengthening of Theorem 5 (see Theorem 16).

1. NOTATION AND TERMINOLOGY

We will generally follow [6]. In particular, every Lindelöf space is regular by definition. A non-empty space is *zero-dimensional* if it is T_1 and it has a base consisting of clopen sets. It is easy to see that every zero-dimensional space is regular (actually, Tychonoff). A space X is *quasi-Lindelöf* if every open cover of X has a countable subcover. A space X is *productively quasi-Lindelöf* if $X \times Y$ is quasi-Lindelöf for every Lindelöf space Y , and it is *powerfully quasi-Lindelöf* if X^ω is quasi-Lindelöf. Given a space X and $U \subseteq X^\omega$, we will say that U is a *basic open set* if $U = \prod_{i \in \omega} V_i$, where each V_i is an open subset of X and $V_i = X$ for all but finitely many i .

The *tightness* $t(X)$ of a space X is the minimum cardinal κ such that whenever $x \in \text{cl}(A)$ for some $A \subseteq X$ then there exists $B \in [A]^{\leq \kappa}$ such that $x \in \text{cl}(B)$. Given a subset A of a space X , define A_α for $\alpha < \omega_1$ by recursion as follows.

- $A_0 = A$.
- $A_{\alpha+1} = \{x \in X : x \text{ is a limit of some sequence of elements of } A_\alpha\}$.
- $A_\gamma = \bigcup_{\alpha < \gamma} A_\alpha$, if γ is a limit ordinal.

A space X is *sequential* if $\text{cl}(A) = \bigcup_{\alpha < \omega_1} A_\alpha$ for every $A \subseteq X$. It is easy to see that every sequential space has countable tightness.

The *Lindelöf number* $\ell(X)$ of a space X is the least cardinal κ such that every open cover of X has a subcover of size at most κ . A family \mathcal{N} of subsets of a space X is a *network* for X if for every $x \in X$ and every neighborhood U of x there exists $N \in \mathcal{N}$ such that $x \in N \subseteq U$. The *network-weight* $nw(X)$ of a space X is the least cardinal κ such that X has a network of size κ . The *weight* $w(X)$ of a space X is the least cardinal κ such that X has a base of size κ . Given a cardinal κ and a set X , a family \mathcal{W} of subsets of X is *point- κ* if $|\{W \in \mathcal{W} : x \in W\}| \leq \kappa$ for every $x \in X$.

We will assume familiarity with the technique of elementary submodels (see for example [4]). As usual, by “elementary submodel” we will really mean “elementary submodel of $H(\theta)$ for a sufficiently large cardinal θ ”. Given an infinite cardinal κ , an elementary submodel M is *κ -closed* if $[M]^{\leq \kappa} \subseteq M$. Given a set S such that $|S| \leq 2^\kappa$, it is easy to construct a κ -closed elementary submodel M such that $S \subseteq M$ and $|M| = 2^\kappa$.

2. ADAPTING A METHOD OF DE LA VEGA

In this section, we adapt to our needs a method that De la Vega developed in [11] (see also [10, Chapter 4]). In fact, Lemma 3 is [11, Lemma 2.2], and the proof of Theorem 4 is inspired by the proof of [11, Lemma 2.3].

Lemma 3 (De la Vega). *Let κ be an infinite cardinal. Assume that (X, τ) is a regular space such that $t(X) \leq \kappa$. Let M be a κ -closed elementary submodel such that $(X, \tau) \in M$, and let $Z = \text{cl}(X \cap M)$. Then, whenever $z_0, z_1 \in Z$ are distinct points, there exist $U_0, U_1 \in \tau \cap M$ such that $z_0 \in U_0$, $z_1 \in U_1$, and $U_0 \cap U_1 = \emptyset$.*

Proof. Fix distinct $z_0, z_1 \in Z$. Since X is Hausdorff, we can fix $U_i \in \tau$ for $i \in 2$ such that $z_i \in U_i$ and $U_0 \cap U_1 = \emptyset$. Since X is regular, we can fix $V_i \in \tau$ for $i \in 2$ such that $z_i \in V_i \subseteq \text{cl}(V_i) \subseteq U_i$. Since $t(X) \leq \kappa$, there exist $A_i \in [V_i \cap M]^{\leq \kappa}$ for $i \in 2$ such that $z_i \in \text{cl}(A_i)$. Notice that each $A_i \in M$ because M is κ -closed. Therefore

$M \models$ There exist $U_0, U_1 \in \tau$ such that $U_0 \cap U_1 = \emptyset$ and $\text{cl}(A_i) \subseteq U_i$ for each i by elementarity, which yields the desired U_0, U_1 . \square

Theorem 4. *Let κ be an infinite cardinal. Assume that (X, τ) is a regular space with $\ell(X) \leq \kappa$ and $t(X) \leq \kappa$. Let M be a κ -closed elementary submodel such that $(X, \tau) \in M$, and let $Z = \text{cl}(X \cap M)$. Then $\{A \cap Z : A \in M\}$ is a network for Z .*

Proof. Fix $z \in Z$ and assume that $z \in O \in \tau$. For each $x \in Z \setminus O$, use Lemma 3 to get $U_x, V_x \in \tau \cap M$ such that $x \in U_x$, $z \in V_x$ and $U_x \cap V_x = \emptyset$. Since $Z \setminus O$ is closed in X , there exists $C \in [Z \setminus O]^{\leq \kappa}$ such that $Z \setminus O \subseteq \bigcup \{U_x : x \in C\}$. Notice that $\mathcal{V} = \{V_x : x \in C\} \in M$ because M is κ -closed. Hence $A = \bigcap \mathcal{V} \in M$ as well. The fact that $z \in A \cap Z \subseteq O$ concludes the proof. \square

3. COUNTABLE TIGHTNESS

In this section, we give an affirmative answer to Question 1 for spaces of countable tightness (see Theorem 7). The main ingredients of the proof are Theorem 4 and Corollary 6. The following result first appeared as [3, Lemma 3.3]. For a proof of a slightly more general result, see Corollary 17.

Theorem 5 (Burton, Tall). *Assume CH. If X is a productively Lindelöf space such that $\ell(X^\omega) \leq \mathfrak{c}$ then X is powerfully Lindelöf.*

Corollary 6. *Assume CH. If X is a productively Lindelöf space with $nw(X) \leq \mathfrak{c}$ then X is powerfully Lindelöf.*

Theorem 7. *Assume CH. Let (X, τ) be a productively Lindelöf space of countable tightness. Then X is powerfully Lindelöf.*

Proof. Fix an open cover \mathcal{U} of X^ω . Let M be an ω -closed elementary submodel such that $\{(X, \tau), \mathcal{U}\} \subseteq M$ and $|M| = \mathfrak{c}$. Let $Z = \text{cl}(X \cap M)$.

First, we will show that $Z^\omega \subseteq \bigcup (\mathcal{U} \cap M)$. So fix $z = (z_i : i \in \omega) \in Z^\omega$. Fix $U \in \mathcal{U}$ such that $z \in U$. Let $V = \prod_{i \in \omega} V_i$ be such that $z \in V \subseteq \text{cl}(V) \subseteq U$, where each $V_i \in \tau$ and $V_i = X$ for all but finitely many i . Given any $i \in \omega$, since $z_i \in \text{cl}(V_i \cap M)$, we can fix $A_i \in [V_i \cap M]^{\leq \omega}$ such that $z_i \in \text{cl}(A_i)$. Using the fact that M is ω -closed, it is easy to see that $A = \prod_{i \in \omega} A_i \in M$. Therefore

$$M \models \text{There exists } U \in \mathcal{U} \text{ such that } \text{cl}(A) \subseteq U$$

by elementarity, which yields $U \in \mathcal{U} \cap M$ such that $z \in U$.

Observe that Z is productively Lindelöf because it is a closed subspace of X . Furthermore, it follows from Theorem 4 that $nw(Z) \leq |M| = \mathfrak{c}$. Therefore Z is powerfully Lindelöf by Corollary 6, hence there exists $\mathcal{V} \in [\mathcal{U} \cap M]^{\leq \omega}$ such that $Z^\omega \subseteq \bigcup \mathcal{V}$. Notice that $\mathcal{V} \in M$ and $X^\omega \cap M = (X \cap M)^\omega \subseteq Z^\omega$ because M is ω -closed. It follows that

$$M \models \mathcal{V} \text{ is a cover of } X^\omega.$$

Therefore, \mathcal{V} is a cover of X^ω by elementarity. \square

4. NEW PROOFS OF RESULTS OF ARKHANGEL'SKII AND BUZYAKOVA

The following two results are [2, Corollary 3.4] and [2, Theorem 4.2]. Recall that a space is *linearly Lindelöf* if it is regular and every open cover of X that is linearly ordered by \subseteq has a countable subcover.

Theorem 8 (Arkhangel'skii, Buzyakova). *Assume CH. Let X be a Tychonoff space of countable tightness such that every open cover of X of size at most ω_1 has a countable subcover. Then X is Lindelöf.*

Theorem 9 (Arkhangel'skii, Buzyakova). *Assume GCH. Let X be a Tychonoff linearly Lindelöf space such that $t(X) < \omega_\omega$. Then X is Lindelöf.*

Using the same techniques as in the previous section, we will give new proofs of the above results. In fact, it is clear that Theorem 8 follows from Theorem 10 and that Theorem 9 follows from Theorem 11. Notice that the assumption “Tychonoff” has been weakened to “regular”.

Theorem 10. *Let κ be an infinite cardinal. Assume that (X, τ) is a regular space such that $t(X) \leq \kappa$ and every open cover of X of size at most 2^κ admits a subcover of size at most κ . Then $\ell(X) \leq \kappa$.*

Proof. Fix an open cover \mathcal{U} of X . Let M be a κ -closed elementary submodel such that $\{(X, \tau), \mathcal{U}\} \subseteq M$ and $|M| = 2^\kappa$. Let $Z = \text{cl}(X \cap M)$. As in the proof of Theorem 7, one can show that $Z \subseteq \bigcup(\mathcal{U} \cap M)$. Since $|\mathcal{U} \cap M| \leq |M| = 2^\kappa$, there exists $\mathcal{V} \in [\mathcal{U} \cap M]^{\leq \kappa}$ such that $Z \subseteq \bigcup \mathcal{V}$. As in the proof of Theorem 7, one sees that \mathcal{V} is a cover of X . \square

Theorem 11. *Assume that $2^\kappa < \omega_\omega$ for every $\kappa < \omega_\omega$. Let (X, τ) be a linearly Lindelöf space such that $t(X) < \omega_\omega$. Then X is Lindelöf.*

Proof. Fix an open cover \mathcal{U} of X . Let $\kappa < \omega_\omega$ be an infinite cardinal such that $t(X) \leq \kappa$. Let M be a κ -closed elementary submodel such that $\{(X, \tau), \mathcal{U}\} \subseteq M$ and $|M| = 2^\kappa$. Let $Z = \text{cl}(X \cap M)$. As in the proof of Theorem 7, one can show that $Z \subseteq \bigcup(\mathcal{U} \cap M)$. Using the fact that X is linearly Lindelöf, it is easy to see that every open cover of X of size less than ω_ω has a countable subcover. Since $|\mathcal{U} \cap M| \leq |M| = 2^\kappa < \omega_\omega$, it follows that there exists $\mathcal{V} \in [\mathcal{U} \cap M]^{\leq \omega}$ such that $Z \subseteq \bigcup \mathcal{V}$. As in the proof of Theorem 7, one sees that \mathcal{V} is a cover of X . \square

5. DROPPING THE SEPARATION AXIOMS

We will use the method of set-valued mappings introduced in [12]. Recall that a *set-valued mapping* from a space X to a space Y is a function $\Phi : X \rightarrow \mathcal{P}(Y)$, where $\mathcal{P}(Y)$ denotes the power-set of Y . A set-valued mapping from X to Y is *compact-valued* if $\Phi(x)$ is a compact subspace of Y for every $x \in X$. A set-valued mapping from X to Y is *upper semi-continuous* if $\{x \in X : \Phi(x) \subseteq V\}$ is open in X for every open subset V of Y . Given any set S , we will identify 2^S with $\mathcal{P}(S)$ through characteristic functions. For $A \in 2^S$, let $A \uparrow = \{B \in 2^S : A \subseteq B\}$.

Lemma 12. *Assume CH. Let X be a productively quasi-Lindelöf space. Then every cover of X^ω of size \mathfrak{c} consisting of basic open sets has a countable subcover.*

Proof. Let $\{U_\alpha : \alpha \in \kappa\}$ be a cover of X^ω consisting of basic open sets, where $\kappa = \mathfrak{c}$. Write $U_\alpha = \prod_{i \in \omega} U_i^\alpha$ for each α , where each U_i^α is an open subset of X and $U_i^\alpha = X$ for all but finitely many i .

Consider the set-valued mapping from X to $2^{\kappa \times \omega}$ obtained by defining

$$\Phi(x) = \{(\alpha, i) \in \kappa \times \omega : x \in U_i^\alpha\} \uparrow$$

for every $x \in X$. Notice that Φ is compact-valued and upper-semicontinuous. It follows that $Y = \bigcup_{x \in X} \Phi(x) \subseteq 2^{\kappa \times \omega}$ is productively Lindelöf. Since $\kappa = \mathfrak{c}$, it is clear that $w(Y) \leq \mathfrak{c}$. Therefore Y is powerfully Lindelöf by Theorem 1.

For each $\alpha \in \kappa$, define

$$V_\alpha = \{(y_i : i \in \omega) \in Y^\omega : (\alpha, i) \in y_i \text{ for every } i \in \omega\}.$$

We claim that $\{V_\alpha : \alpha \in \kappa\}$ is an open cover of Y^ω . First we will prove that $V_\alpha = \{(y_i : i \in \omega) \in Y^\omega : (\alpha, i) \in y_i \text{ for every } i \in \omega \text{ such that } U_i^\alpha \neq X\}$. Notice that this implies that each V_α is open. The inclusion \subseteq is obvious. In order to prove the other inclusion, fix $y = (y_i : i \in \omega) \in Y^\omega$ such that $(\alpha, i) \in y_i$ for every $i \in \omega$ such that $U_i^\alpha \neq X$. By the definition of Y , there exists $(x_i : i \in \omega) \in X^\omega$ such that $y_i \supseteq \{(\beta, j) \in \kappa \times \omega : x_i \in U_j^\beta\}$ for each i . We have to show that $(\alpha, i) \in y_i$ for each i . So fix $i \in \omega$. If $U_i^\alpha \neq X$ then $(\alpha, i) \in y_i$ by assumption. On the other hand, if $U_i^\alpha = X$ then $(\alpha, i) \in \{(\beta, j) \in \kappa \times \omega : x_i \in U_j^\beta\} \subseteq y_i$. Next, we will show that $\{V_\alpha : \alpha \in \kappa\}$ covers Y^ω . So fix $y = (y_i : i \in \omega) \in Y^\omega$. By the definition of Y , there exists $x = (x_i : i \in \omega) \in X^\omega$ such that $y_i \supseteq \{(\beta, j) \in \kappa \times \omega : x_i \in U_j^\beta\}$ for each i . Let $\alpha \in \kappa$ be such that $x \in U_\alpha$. It is clear that $y \in V_\alpha$.

To conclude the proof, assume that $S \subseteq \kappa$ is such that $\{V_\alpha : \alpha \in S\}$ covers Y^ω . It will be enough to show that $\{U_\alpha : \alpha \in S\}$ covers X^ω . So fix $x = (x_i : i \in \omega) \in X^\omega$. Define $y_i = \{(\beta, j) \in \kappa \times \omega : x_i \in U_j^\beta\}$ for each i , and notice that each $y_i \in Y$. Since $y = (y_i : i \in \omega) \in Y^\omega$, there exists $\alpha \in S$ such that $y \in V_\alpha$. It follows from the definitions of V_α and y_i that $x \in U_\alpha$. \square

Notice that the proof of Lemma 12 also yields the following result. Corollary 14 shows that separation axioms are irrelevant to Question 1. The fact that separation axioms are irrelevant to the other, more famous, question of Michael (whether ω^ω is productively Lindelöf) was proved by Duanmu, Tall and Zdomskyy using the same methods (see [5, Lemma 1]).

Theorem 13. *Let κ be an infinite cardinal. If there exists a productively quasi-Lindelöf space X with $w(X) \leq \kappa$ that is not powerfully quasi-Lindelöf, then there exists a zero-dimensional productively Lindelöf space Y with $w(Y) \leq \kappa$ that is not powerfully Lindelöf.*

Corollary 14. *The following are equivalent.*

- Every productively quasi-Lindelöf space is powerfully quasi-Lindelöf.
- Every productively Lindelöf space is powerfully Lindelöf.
- Every zero-dimensional productively Lindelöf space is powerfully Lindelöf.

6. POINT- \mathfrak{c} FAMILIES

In this section, we give an affirmative answer to Question 1 for one more class of spaces (see Theorem 16). The main ingredients of the proof are Lemma 12 and Lemma 15.

Lemma 15. *Let X be a set, and let κ be an infinite cardinal. Assume that \mathcal{W} is a point- 2^κ family of subsets of X . Let M be a κ -closed elementary submodel such that $\{X, \mathcal{W}\} \subseteq M$. If $W \in \mathcal{W}$ and $W \cap M \neq \emptyset$ then $W \in M$.*

Proof. Define $\mathcal{W}_x = \{W \in \mathcal{W} : x \in W\}$ for $x \in X$, and notice that $|\mathcal{W}_x| \leq 2^\kappa$ for every $x \in X$. Now fix $W \in \mathcal{W}$ such that $W \cap M \neq \emptyset$. Let $z \in W \cap M$, and observe that $\mathcal{W}_z \in M$. By elementarity,

$$M \models \text{There exists a surjection } f : \mathcal{P}(\kappa) \longrightarrow \mathcal{W}_z.$$

Furthermore, $\mathcal{P}(\kappa) \subseteq M$ because M is κ -closed. Therefore $\mathcal{W}_z \subseteq M$, and in particular $W \in M$. \square

Theorem 16. *Assume CH. Let (X, τ) be a productively quasi-Lindelöf space such that every open cover of X^ω has a point- \mathfrak{c} refinement consisting of basic open sets. Then X is powerfully quasi-Lindelöf.*

Proof. It will be enough to show that every point- \mathfrak{c} cover of X^ω consisting of basic open sets has a countable subcover. So fix such a cover \mathcal{W} . Let M be an ω -closed elementary submodel such that $\{(X, \tau), \mathcal{W}\} \subseteq M$ and $|M| = \mathfrak{c}$. Let $Z = \text{cl}(X \cap M)$.

We claim that $Z^\omega \subseteq \bigcup(\mathcal{W} \cap M)$. Fix $z \in Z^\omega$. Let $W \in \mathcal{W}$ be such that $z \in W$. Using the fact that M is ω -closed, it is easy to check that $Z^\omega = \text{cl}(X^\omega \cap M)$. Therefore $W \cap M \neq \emptyset$. Hence $W \in M$ by Lemma 15, which proves our claim.

Since $|\mathcal{W} \cap M| \leq \mathfrak{c}$, it follows from Lemma 12 that there exists $\mathcal{V} \in [\mathcal{W} \cap M]^{\leq \omega}$ such that $Z^\omega \subseteq \bigcup \mathcal{V}$. Now proceed as in the proof of Theorem 7. \square

The following corollary shows that Theorem 16 might be viewed as a strengthening of Theorem 5.

Corollary 17. *Assume CH. Let X be a productively quasi-Lindelöf space such that $\ell(X^\omega) \leq \mathfrak{c}$. Then X is powerfully quasi-Lindelöf.*

As a further corollary of Theorem 16 one obtains that, under CH, every productively Lindelöf space with a point- \mathfrak{c} base is powerfully Lindelöf, which is a strengthening of Theorem 1. However, as Corollary 19 shows, the improvement is illusory. Although we could not find it in the literature, we feel that Theorem 18 might already be known. In fact, it is inspired by the classical result of Miščenko stating that every compact space with a point-countable base has a countable base (see [8] or [6, Exercise 3.12.23(f)]), which can be proved using a similar argument (let M be countable instead of κ -closed).

Theorem 18. *Let κ be an infinite cardinal. Assume that (X, τ) is a T_1 space such that $\ell(X) \leq \kappa$ and X has a point- 2^κ base. Then $w(X) \leq 2^\kappa$.*

Proof. Fix a point- 2^κ base \mathcal{B} for X . Let M be a κ -closed elementary submodel such that $\{(X, \tau), \mathcal{B}\} \subseteq M$ and $|M| = 2^\kappa$. Define $\mathcal{B}_x = \{B \in \mathcal{B} : x \in B\}$ for $x \in X$, and notice that $|\mathcal{B}_x| \leq 2^\kappa$ for every $x \in X$. We claim that $X \cap M$ is dense in X . Since this implies $\mathcal{B} = \bigcup_{x \in X \cap M} \mathcal{B}_x$, hence $|\mathcal{B}| \leq 2^\kappa$, this will conclude the proof.

Assume, in order to get a contradiction, that $z \in X \setminus \text{cl}(X \cap M)$. Define

$$\mathcal{U} = \{B \in \mathcal{B} : B \cap M \neq \emptyset \text{ and } z \notin B\},$$

and notice that $\mathcal{U} \subseteq M$ by Lemma 15. Using the fact that $\{z\}$ is closed, one sees that \mathcal{U} is a cover of $\text{cl}(X \cap M)$. Therefore, there exists $\mathcal{V} \in [\mathcal{U}]^{\leq \kappa}$ such that \mathcal{V} is a cover of $\text{cl}(X \cap M)$. Observe that $\mathcal{V} \in M$ because $\mathcal{V} \subseteq \mathcal{U} \subseteq M$ and M is κ -closed, hence

$$M \models \mathcal{V} \text{ is a cover of } X.$$

By elementarity, it follows that \mathcal{V} is a cover of X , contradicting our choice of z . \square

Corollary 19. *Let X be a Lindelöf space with a point- \mathfrak{c} base. Then $w(X) \leq \mathfrak{c}$.*

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